Direct Proof – Universal Statements

Lecture 13
Section 4.1

Robb T. Koether

Hampden-Sydney College

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Proving Universal Statements

- A universal statement is of the form

$$\forall x \in D, P(x).$$

- Use the method of generalizing from the generic particular.
  - Select an *arbitrary* $x \in D$ (generic particular).
  - Assume nothing about x that is not inherent to all elements of the set D.
  - Show that $P(x)$ is true.

The Generic Particular

Theorem

The sum of two odd integers is an even integer.

- Let n and m be odd integers.
- Then $n = 2s + 1$ and $m = 2t + 1$ for some integers s and t.
- Then

$$\begin{aligned} n + m &= (2s + 1) + (2t + 1) \\ &= 2s + 2t + 2 \\ &= 2(s + t + 1). \end{aligned}$$

- We know that $s + t + 1$ is an integer, so $n + m$ is an even integer.

Proving Existential Statements

- A existential statement is of the form

$$\exists x \in D, P(x).$$

- We must either
  - Choose a specific $x \in D$ and show that $P(x)$ is true, or
  - Argue indirectly that such an x must exist.

Proving Existential Statements

Theorem

There exist integers a and b such that

$$a^2 + b^2 = 1000.$$

- Let $a = 18$ and $b = 26$.
- Then

$$\begin{aligned} a^2 + b^2 &= 18^2 + 26^2 \\ &= 324 + 676 \\ &= 1000. \end{aligned}$$

Proving Statements with Mixed Quantifiers

- Many theorems are of the form

$$\forall x \in D_1, \exists y \in D_2, P(x, y).$$

- Many other theorems are of the form

$$\exists x \in D_1, \forall y \in D_2, P(x, y).$$

Proving Statements with Mixed Quantifiers

- For the form
$$\forall x \in D_1, \exists y \in D_2, P(x, y),$$
  - Choose a generic particular $x \in D_1$.
  - Then choose a specific $y \in D_2$ *in terms of* x and show that $P(x, y)$ is true.

Proving Statements with Mixed Quantifiers

- For the form

$$\exists x \in D_1, \forall y \in D_2, P(x, y),$$

  - Choose a specific $x \in D_1$.
  - Then choose a generic particular $y \in D_2$ and show that $P(x, y)$ is true.
- If you must argue indirectly that such an x exists, then choose the generic particular y *in terms of* x.

Proving Statements with Mixed Quantifiers

Theorem

For every real number a and for every real number $b > a$, there exists a real number c such that

$$a < c < b.$$

- We could restate the theorem as

$$\forall a \in \mathbb{R}, \forall b \in \mathbb{R} \text{ with } b > a, \exists c \in \mathbb{R}, a < c < b.$$

Proving Statements with Mixed Quantifiers

- Let a and b be real numbers, with $b > a$.
- Let $c = \dfrac{a+b}{2}$.
- Then

$$\begin{aligned} a &< b \\ a + b &< 2b \\ \frac{a+b}{2} &< b \\ c &< b. \end{aligned}$$

Proving Statements with Mixed Quantifiers

- Similarly,

$$\begin{aligned} a &< b \\ 2a &< a + b \\ a &< \frac{a+b}{2} \\ a &< c. \end{aligned}$$

- Therefore, $a < c < b$.

Proving Statements with Mixed Quantifiers

Theorem

There exists a real number c such that for every real number $x > c$,

$$x^2 > 100x + 1000.$$

- We could restate the theorem as

$$\exists c \in \mathbb{R}, \forall x \in \mathbb{R} \text{ with } x > c, x^2 > 100x + 1000,$$

or

$$\exists c \in \mathbb{R}, \forall x \in \mathbb{R}, x > c \to x^2 > 100x + 1000.$$

Proving Statements with Mixed Quantifiers

- Let $c = 110$.
- Let x be any real number greater than 110.
- Then

$$\begin{aligned} x &> 110 \\ x - 50 &> 60 \\ (x-50)^2 &> 60^2 \\ x^2 - 100x + 2500 &> 3600 \\ x^2 &> 100x + 1100 \\ x^2 &> 100x + 1000. \end{aligned}$$

Proving Statements with Mixed Quantifiers

Theorem

For every real number m and for every real number b, there exists a real number c such that for every real number $x > c$,

$$x^2 > mx + b.$$

- We could restate the theorem as

$$\forall m \in \mathbb{R}, \forall b \in \mathbb{R}, \exists c \in \mathbb{R}, \forall x \in \mathbb{R}, x > c \rightarrow x^2 > mx + b.$$

Theorem

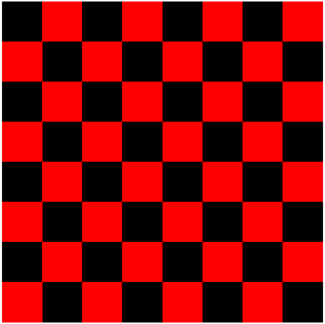
Let x be a real number. If for all real numbers y, we have $xy = 0$, then $x = 0$.

- The form of this statement is

$$\forall x \in \mathbb{R}, ((\forall y \in \mathbb{R}, xy = 0) \to x = 0).$$

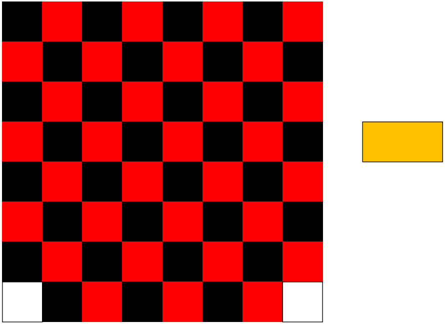
- How do we prove this?
- What is its negation?

Proving an Existential Statement



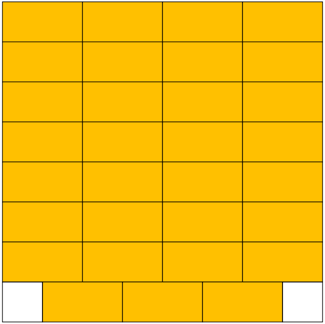
A checkerboard

Proving an Existential Statement



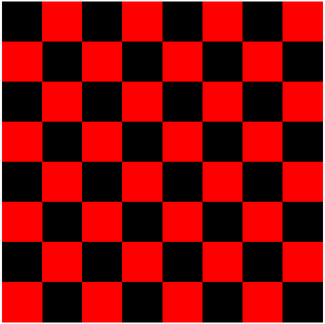
Remove two squares and cover board with 1×2 blocks

Proving an Existential Statement



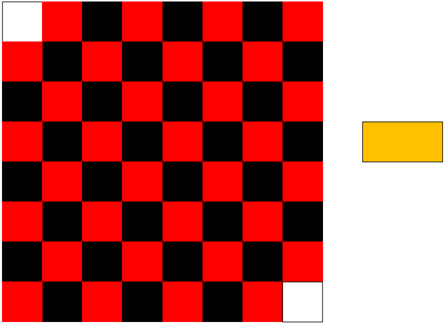
It can be done

Disproving an Existential Statement



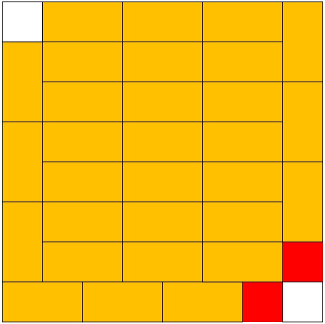
A checkerboard

Disproving an Existential Statement



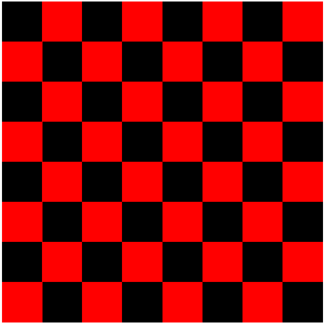
Remove two squares and cover board with 1×2 blocks

Disproving an Existential Statement



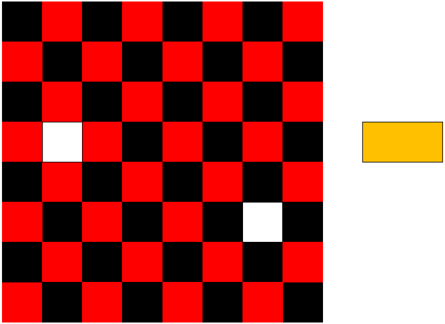
It cannot be done

A Universal and Existential Statement



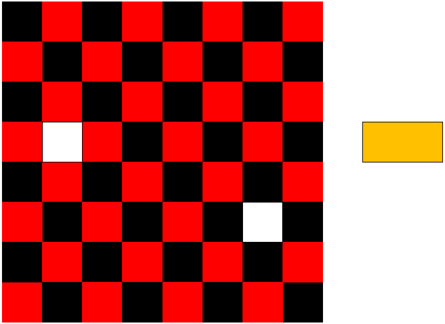
Remove any two squares of *opposite* color

A Universal and Existential Statement



Can the board necessarily be covered?

A Universal and Existential Statement



Is that true for any $2n \times 2n$ board?

Assignment

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- Read Section 4.1, pages 145 - 160.
- Exercises 20, 21, 25, 28, 30, 37, 52, 53, 55, 58, page 161.